Indukované zobrazení, diffeomorfismy, toky a Lieova derivace

Indukované zobrazení na tenzorech

Diferenciál zobrazení

Indukované zobrazení na polích

Tok a jeho generátor

Lieova derivace

Interpretace Lieovy závorky

Indukované zobrazení na tenzorech

připomínka:

hladké zobr $\phi : M \to N$

souř. vyjádření hladké $\bar{\phi} = y \circ \phi \circ x^{-1}$ hladké

kde (U, x) mapa na M a (V, y) mapa na N

difeomorfismus $\phi : M \to N$ či $\phi : M \to M$

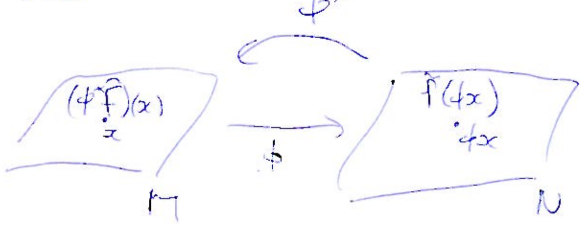
existuje hladké inverz. zobr ϕ^{-1}

indukované zobr. pro $\phi : M \to N$

pull-back (stáhnutí) fce

$\phi^* : \mathcal{F}N \to \mathcal{F}M \qquad \tilde{f} \to \phi^* \tilde{f}$

$(\phi^* \tilde{f})(x) = \tilde{f}(\phi x)$

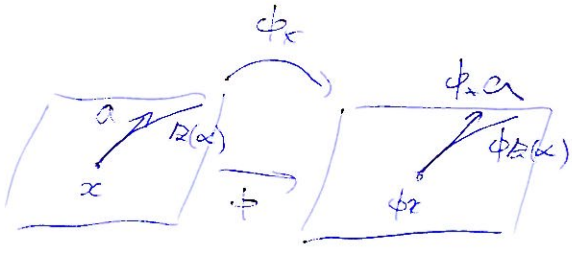


push-forward (přenos) vektoru

$\phi_* : \mathbf{T}_x M \to \mathbf{T}_{\phi x} N$

$\phi_* \dfrac{Dz}{d\alpha}\Big|_{\alpha=0} = \dfrac{D\phi z}{d\alpha}\Big|_{\alpha=0}$.

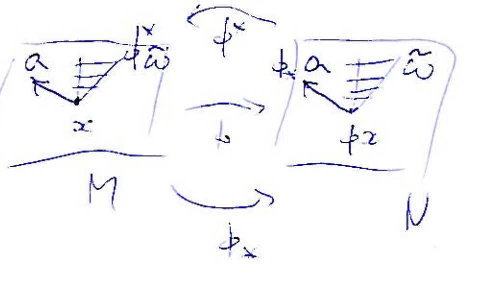
$z(\alpha)$ křivka v M s teč. vekt a



pull-back (stáhnutí) 1-formy

$\phi^* : \mathbf{T}^*_{\phi x} N \to \mathbf{T}^*_x M$

$(\phi^* \tilde{\omega}) \cdot a|_x = \tilde{\omega} \cdot (\phi_* a)|_{\phi x}$



platí

$(\phi_* a)|_{\phi x} [\tilde{f}] = a|_x [\phi^* \tilde{f}]$

$(\phi_* a)|_{\phi x} [\tilde{f}] = \dfrac{d}{d\alpha} \tilde{f}(\phi z(\alpha))\Big|_{\alpha=0} = \dfrac{d}{d\alpha} (\phi^* \tilde{f})(z(\alpha))\Big|_{\alpha=0} = a|_x [\phi^* \tilde{f}] \qquad a = \dot{z}$

$\phi^* (d\tilde{f}|_{\phi x}) = d(\phi^* \tilde{f})|_x$

$a|_x \cdot \phi^*(d\tilde{f}|_{\phi x}) = (\phi_* a)|_{\phi x} \cdot d\tilde{f}|_{\phi x} = (\phi_* a)|_{\phi x} [\tilde{f}] = a|_x [\phi^* \tilde{f}] = a|_x \cdot d(\phi^* \tilde{f})|_x$

$\Rightarrow \phi^*(d\tilde{f}|_{\phi x}) = d(\phi^* \tilde{f})|_x$

linearita indukovaných zobr.

$$\phi_*(a+rb) = \phi_* a + r\phi_* b \qquad r\in\mathbb{R}$$

$$\Leftarrow (\phi_*(a+rb))[\hat f] = (a+rb)[\phi^*\hat f] = a[\phi^*\hat f] + r\, b[\phi^*\hat f] = (\phi_* a)[\hat f] + r(\phi_* b)[\hat f]$$

$$\phi^*(\tilde\alpha + r\tilde\beta) = \phi^*\tilde\alpha + r\,\phi^*\tilde\beta$$

$$\Leftarrow \phi^*(\tilde\alpha + r\tilde\beta)\cdot u = (\tilde\alpha + r\tilde\beta)\cdot\phi_* u = \tilde\alpha\cdot\phi_* u + r\,\tilde\beta\,\phi_* u = (\phi^*\tilde\alpha + r\,\phi^*\tilde\beta)\cdot u$$

zobecnění na tenzory

komutace s tenz. násobením $\otimes$ $\rightarrow$ rozšíření

$$\phi_* : \mathbb{T}_x{}^p_0 M \rightarrow \mathbb{T}_{\phi x}{}^p_0 N$$

$$\phi^* : \mathbb{T}_{\phi x}{}^0_p N \rightarrow \mathbb{T}_x{}^0_p M$$

nelze definovat na obecných tenz $\mathbb{T}{}^p_q$ pro obecné ϕ

indukt. zobr. pro difeomorf.

podmínka $\phi^* = \phi^{-1}{}_*$ $\quad \phi_* = \phi^{-1*}$ umožňuje rozšíření

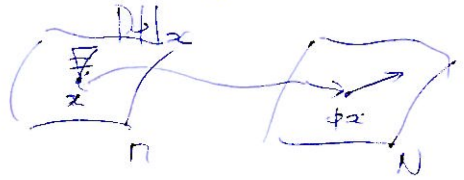
$$\phi_* : \mathbb{T}_x{}^p_q M \rightarrow \mathbb{T}_{\phi x}{}^p_q N$$

$$\phi^* : \mathbb{T}_{\phi x}{}^p_q N \rightarrow \mathbb{T}_x{}^p_q M$$

Diferenciál zobrazení

díky linearitě indukovaného zobr. lze repre z. tenzorově

$$D\phi|_x \in T_x^* M \otimes T_{\phi x} N$$



$$\underset{T_{\phi x} N}{(\phi_* a)^{\tilde{n}}} = D^{\tilde{n}}_{m} \phi|_x \underset{T_x M}{a^{m}}$$

$$\underset{T^*_x M}{(\phi^* \tilde{\alpha})_{m}} = D^{\tilde{n}}_{m} \phi|_x \underset{T^*_{\phi x} N}{\tilde{\alpha}_{\tilde{n}}}$$

vyjádření v souřadnicích

(U, x) mapa na M $\quad$ (V, y) mapa na N

$$\Phi = y \circ \phi \circ x^{-1}$$

$$D\phi = \frac{\partial \Phi^{\tilde{a}}}{\partial x^{a}}\, dx^{a}\, \frac{\partial}{\partial y^{\tilde{a}}}$$

plyne z aplikace $\phi_* \frac{\partial}{\partial x^a}$ a $\phi^* dy^{\tilde{a}}$

složení zobrazení

$$(\phi \circ \psi)_* = \phi_* \circ \psi_* \quad \Rightarrow \quad D(\phi\circ\psi)|_x = D\phi|_{\psi x} \cdot D\psi|_x$$

působení na tenzorech

ϕ difeomorf.

$$\phi^{-1} \circ \phi = \mathrm{id}_M \quad \phi \circ \phi^{-1} = \mathrm{id}_N \quad \Rightarrow \quad D\phi \cdot D\phi^{-1} = \delta_N \quad D\phi^{-1} \cdot D\phi = \delta_M$$

$$(\phi_* T)^{\tilde{a}_1 \tilde{a}_2 \dots}_{\tilde{b}_1 \tilde{b}_2 \dots} = D^{\tilde{a}_1}_{m_1}\phi\, D^{\tilde{a}_2}_{m_2}\phi \dots D^{n_1}_{\tilde{b}_1}\phi^{-1}\, D^{n_2}_{\tilde{b}_2}\phi^{-1} \dots T^{m_1 m_2 \dots}_{n_1 n_2 \dots}$$

speciální případy

1-dim varieta M lze ztotožnit s $\mathbb{R}$ volbou souř. τ

TM je 1. dim – ztotož. s $\mathbb{R}$ pomocí $\frac{\partial}{\partial\tau}$

T^*M je 1 dim. – ztotož. s $\mathbb{R}$ pomocí $d\tau$

funkce na M

$f : M \to N = \mathbb{R}$ $\qquad \bar{f} = \tau(f) : M \to \mathbb{R}$

$Df = d\bar{f}\,\frac{\partial}{\partial\tau} = \bar{f}_{,i}\, dx^i \frac{\partial}{\partial\tau}$

lze ztotožnit Df s $d\bar{f}$

křivka v N

$z : M = \mathbb{R} \to N \qquad \bar{z} \circ \tau = z \qquad \bar{z} : \mathbb{R} \to N$

$Dz = d\tau\,\frac{D\bar{z}}{d\tau} = \frac{d\bar{z}^i}{d\tau}\, d\tau\, \frac{\partial}{\partial y^i} \qquad \bar{z}^i = y^i(\bar{z})$

lze ztotožnit Dz a $\frac{D\bar{z}}{d\tau} \equiv \dot{\bar{z}}$

Indukované zobrazení na polích

funkce $\phi^* : \mathcal{F}N \to \mathcal{F}M$

$$(\phi^* \tilde{f})(x) = \tilde{f}(\phi x)$$

formy $\phi^* \quad T^0_p N \to T^0_p M$

$$\underbrace{(\phi^* \tilde{\omega})(x)}_{T_x{}^0_p M} = \phi^* (\underbrace{\tilde{\omega}(\phi x)}_{T_{\phi x}{}^0_p N})$$

platí

$$\phi^* df = d \phi^* f$$

plyne z relace v bodě

nelze definovat push-forward vekt pole

$$\underbrace{(\phi_* a)(\tilde{x})}_{T_{\tilde{x}} N} = \phi_* (\underbrace{a(\phi^{-1} \tilde{x})}_{T_{\phi^{-1}\tilde{x}} M}) \qquad \phi^{-1} \text{ není definováno!!}$$

potřeba inverze ϕ

difeomorfismy

push-forward:

$$(\phi_* A)(\tilde{x}) = \phi_* (A(\phi^{-1} \tilde{x}))$$

$$(\phi_* A)(\phi x) = \phi_* (A(x))$$

pullback

$$(\phi^* \tilde{A}) = \phi_*^{-1} A$$

$$(\phi^* \tilde{A})(x) = \phi^* \tilde{A}(\phi x)$$

konzistentní s definicí ϕ^* pro formy

vlastnosti

$$\phi_* (A + rB) = \phi_* A + r \phi_* B$$

$$\phi_* (A \otimes B) = (\phi_* A) \otimes \phi_* B$$

$$\phi_* \mathsf{C} A = \mathsf{C} \phi_* A$$

vlastnosti

$$a[\phi^* \tilde{f}] = \phi^*((\phi_* a)[\hat{f}])$$

$$\phi_*(a[\phi^* \hat{f}]) = (\phi_* a)[\hat{f}]$$

$$\phi_*(a[f]) = \phi_* a[\phi_* f]$$

ekvivalentní

dk:

$$(a[\phi^* \hat{f}])(x) = a(x)[\phi^* \hat{f}] = (\phi_*(a(x)))[\tilde{f}] \overset{\text{difeo}}{=}$$

$$= ((\phi_* a)(\phi x))[\hat{f}] = ((\phi_* a)[\hat{f}])\Big|_{\phi x} = \phi^*((\phi_* a)[\hat{f}])(x)$$

Lieova závorka

$$\phi_*[a,b] = [\phi_* a, \phi_* b]$$

dk:

$$\phi^*((\phi_*[a,b])[\hat{f}]) = [a,b][\phi^* \hat{f}] = a[b[\phi^* \hat{f}]] - b[a[\phi^* \hat{f}]]$$

$$= a[\phi^*((\phi_* b)[\hat{f}])] - b[\phi^*((\phi_* a)[\hat{f}])]$$

$$= \phi^*(\phi_* a[(\phi_* b)[\hat{f}]] - (\phi_* b)[(\phi_* a)[\hat{f}]])$$

$$= \phi^*([\phi_* a, \phi_* b][\hat{f}])$$